Criticality and oscillatory behavior in non-Markovian contact process

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A non-Markovian generalization of a one-dimensional contact process is being introduced in which every particle has an age and will be annihilated at its maximum age τ. There is an absorbing state phase transition which is controlled by this parameter. The model can demonstrate oscillatory behavior in its approach to the stationary state. These oscillations are also present in the mean-field approximation, which is a first-order differential equation with time delay. Studying dynamical critical exponents suggests that the model belongs to the direct percolation universality class.

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I. INTRODUCTION

Studying phase transitions in the systems far from equilibrium has been a topic of growing interest in recent years [1,2]. In particular, systems with absorbing states which cannot evolve further once they are trapped in one such state have been an interesting subject of research. Various models with one or more absorbing states have been studied which belong to a few universality classes and mostly to that of directed percolation (DP). According to the DP conjecture [3,4], every phase transition in a system with a single absorbing state having short-range interactions with no special symmetry or quenched disorder [5] belongs to the DP class. There have been examples of absorbing state phase transitions without some of the DP conjecture conditions that still belong to the DP class, such as systems with an infinite number of absorbing states [6,7]. There have also been other universality classes for parity conserving systems [8] and, as recently proposed, for systems with infinitely many absorbing states coupled to a conserved field [9,10].

Although the Markovian property has been implicitly accepted in these models it is not essential for a nonequilibrium phase transition. Usually, adding some kind of memory to the system, such that the system should refer to its history in order to define its future, gives rise to some new interesting behaviors that are absent in Markovian systems [11,12]. However, properties of non-Markovian nonequilibrium phase transitions have not been studied.

In this paper a non-Markovian variant of the contact process (CP) [13] is introduced and the critical behavior is investigated. Standard CP in its continuous time version is a lattice model in which every empty site is occupied by a particle with rate $\lambda n/z$ and every particle is removed with rate one, where z is the coordination number, n is the number of occupied nearest neighbors, and λ is a positive parameter controlling the creation rate. The system has a second-order critical point at $\lambda_c = 3.2978$ and it belongs to the DP class [2].

In this model I introduce a *memory* for each particle; every particle knows when it has been created. Like standard CP, every site is being occupied by a particle at a rate proportional to the number of its occupied nearest neighbors, while every existing particle will die exactly at age τ. For large values of τ the particles live enough to reproduce plenty of others and the system can remain in its active state. As τ is decreased, each particle has less time to create other particles and for $\tau < \tau_c$ the system will be trapped in its absorbing state with probability 1, where there is no existing particle and no new particles can be born.

Attributing age to the particles in the CP model has been suggested earlier [14], but not in a way that leads to a non-Markovian model; although some interesting alterations in the dynamical behavior of the system have been observed.

Non-Markovian property gives rise to some oscillations in the density of particles. These oscillations are also supported by the mean-field approximation. Because of the non-Markovian property of the system, there is a delay parameter in the mean-field equation and this reproduces the oscillatory solutions observed in the simulations. By the mean-field approach, existence of the phase transition is justified and a critical age can be found, but like standard CP the critical behavior is not described completely.

In this paper, the critical behavior of the model is studied using the time-dependent Monte Carlo method. The critical dynamical exponents has been calculated, and shown to be in good agreement with those of DP.

II. MODEL

The model is defined on a one-dimensional lattice and with continuous time. Every site is either empty or occupied by a single particle. There is a chance for a vacant site to be occupied provided there are occupied sites in its nearest neighborhood. A new particle is born in an empty site with rate $n/2$ (n is the number of occupied nearest neighbors). Every particle will die exactly at time τ after its birth.

Obviously there must be a phase transition in the system with density of the particles as the order parameter. For small values of τ, particles die fast and eventually the system is trapped in its absorbing state. For large τ's, particles have a large lifetime and sufficiently reproduce others to keep the system active. Figure 1 shows a single cluster in a realization of the model for $\tau = 3.5$ and up to $t = 100$. As can be seen,

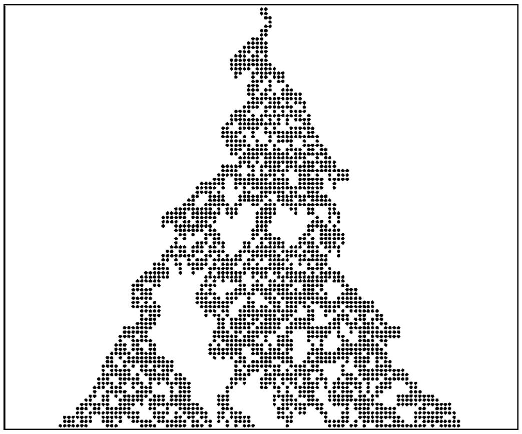

FIG. 1. A typical space-time cluster started with a single particle in the origin for $\tau=3.5$ and up to $t=100$.

every particle has a definite lifetime and here the system is in its active state.

Like CP, here the creation process is not history dependent, however, the death process is, and thus we have a non-Markovian process. To find out that how many particles are removed at each moment, we should know how many particles had been created at time τ earlier. Therefore, a knowledge of only the present state of the system is not enough to find out future states.

III. MEAN-FIELD EQUATION

Let $\sigma_1(x)$ and $\sigma_0(x)$ denote the state of site x, and ρ be the density of particles. $\sigma_1(x)$ is 1 if the site is occupied and 0 if it is vacant, and we have

$$\rho=\langle\sigma_1(x)\rangle_x=1-\langle\sigma_0(x)\rangle_x. \tag{1}$$

Therefore, the rate of reproduction is

$$\langle\sigma_1(x)\sigma_0(x+1)\rangle_x \tag{2}$$

and it can be written in terms of the density-vacancy correlation function (the correlation of the occupied and vacant sites)

$$C_{10}(\delta)=\frac{\langle\sigma_1(x)\sigma_0(x+\delta)\rangle_x-\rho(1-\rho)}{\rho(1-\rho)}. \tag{3}$$

Hence the rate of reproduction is

$$r\rho_t(1-\rho_t), \tag{4}$$

where

$$r=1+C_{10}(\delta=1). \tag{5}$$

Obviously $C_{10}(\delta=1)$ is negative (a particle reduces the chance of its nearest neighbors to be vacant), and thus $r<1$. In the mean-field approximation the correlation is neglected and we put $r=1$. Therefore, the mean-field equation will be

$$\frac{d\rho_t}{dt}=\rho_t(1-\rho_t)-\rho_{t-\tau}(1-\rho_{t-\tau}) \quad (t>\tau), \tag{6}$$

where the second term is the rate of annihilation at time t, equal to the rate of creation at time $t-\tau$. This equation is true for $t>\tau$. I assume that all existing particles at $t=0$ gradually die during the time interval $(0,\tau)$. So for $t<\tau$ we have

$$\frac{d\rho_t}{dt}=\rho_t(1-\rho_t)-\rho_0/\tau \quad (t\leqslant\tau). \tag{7}$$

This equation can be rewritten in the integral form. First by integrating Eq. (7),

$$\rho_t=\int_0^t\rho_{t'}(1-\rho_{t'})dt'-\rho_0 t/\tau+\rho_0 \quad (t\leqslant\tau) \tag{8}$$

especially for $t=\tau$,

$$\rho_{t=\tau}=\int_0^\tau\rho_{t'}(1-\rho_{t'})dt'. \tag{9}$$

By integrating Eq. (6) and making use of Eq. (9) we find

$$\rho_t=\int_{t-\tau}^t\rho_{t'}(1-\rho_{t'})dt' \quad (t>\tau). \tag{10}$$

It is a definite integral with a time-dependent lower and upper limit. So although the integrand is non-negative, $\rho(t)$ may have a nonmonotonic behavior.

Finding stationary density is not possible in the differential equation. Setting $d\rho_t/dt=0$ leads to nothing more than $\rho_t=1-\rho_{t-\tau}$ or $\rho_t=\rho_{t-\tau}$. The former is irrelevant in the steady state and the latter is correct for every value of $\overline{\rho}$. However, by setting $\rho_t=\rho_{t'}=\overline{\rho}$ in the integral equation [Eq. (10)], it turns out that

$$\overline{\rho}=0 \tag{11}$$

or

$$\overline{\rho}=1-1/\tau. \tag{12}$$

Thus there is a phase transition at $\tau=\tau_c=1$.

IV. OSCILLATIONS

It is observed that the density of particles ρ undergoes damped oscillations while approaching the stationary state. The solid line in Fig. 2 shows one such oscillatory evolution of ρ. Simulations are done in a lattice of 10 000 sites with periodic boundary condition. The plotted curves are averaged over 100 realizations for $\tau=7$. A period of oscillation is slightly greater than τ. These kinds of oscillations are present for all values of τ, but they are weaker for smaller τ.

The oscillatory behavior can be understood by paying attention to the history-dependence feature of the model. Since every particle dies at age τ, the time evolution at time t is coupled to the state of the system at time $t-\tau$. A high cre-

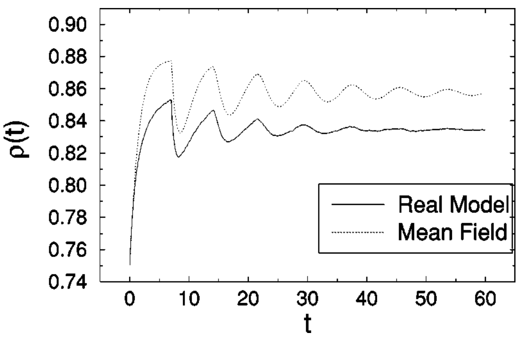

FIG. 2. Density of particles vs time for non-Markovian CP in the real model (solid line) and in the mean-field approximation (dashed line) for $\tau=7$ and $\rho_0=0.75$.

ation rate at time t is equivalent to a high annihilation rate at time $t+\tau$. So, an increase in the density at time t can lead to a decrease in it at some time later and naturally the period of the oscillations is of order τ.

These oscillations are also supported by the mean-field approximation. The mean-field equation is a delayed nonlinear first-order differential equation that is able to have oscillatory solutions which do not exist in ordinary first-order differential equations. Figure 2 (dashed line) shows one of these oscillatory solutions for $\tau=7$. Periods of oscillations in the real model and the mean-field approximation are the same.

The delay time in the mean-field differential equation can be eliminated by a Taylor expansion of $\rho_{t-\tau}$. It basically contains derivatives up to infinite order. In fact, here we have an infinite-order differential equation which is naturally able to demonstrate many complex behaviors. As in simulations, oscillatory behavior is sensitive to changes in the value of τ. It disappears for small enough values of τ and it is less damped for larger τ's.

V. CRITICAL BEHAVIOR

In this section I will present the results concerning the critical behavior of the system obtained from simulating the model. Simulation is made using the time-dependent Monte Carlo method [15]. In this method, the simulation is started with the system in a state very close to the absorbing state, i.e., all the sites are vacant except the one in the origin which is occupied. The age of this single particle is initially set to 0. The sites are updated parallel and after every time increment. All existing particles become older by that amount. They die after growing up to age τ.

I measured the average population of particles $N(t)$, averaged over all realizations, $P(t)$, the probability of not entering the absorbing state up to time t, and $R^2(t)$ the mean-square spreading distance. As a result of the scaling hypothesis [15], at criticality, these quantities should scale algebraically as

$$N(t)\sim t^{\eta}, \tag{13}$$

$$P(t)\sim t^{-\delta}, \tag{14}$$

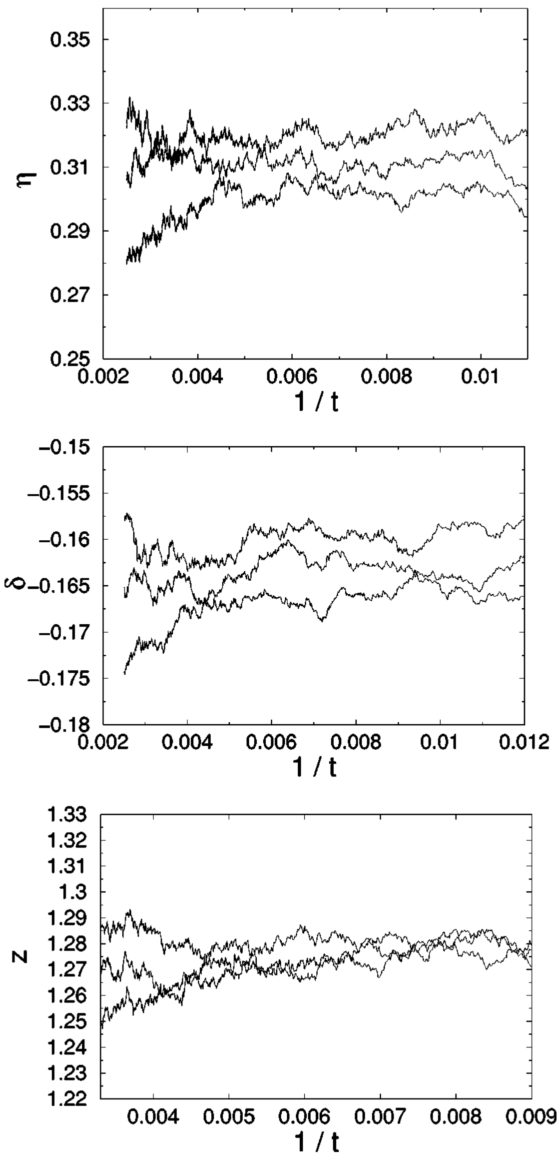

FIG. 3. Dynamical critical exponents as functions of $1/t$. Different curves in each panel correspond, from top to bottom, to $\tau=3.06$, 3.07, and 3.08, respectively.

$$R^2(t)\sim t^{z}. \tag{15}$$

So at criticality the log-log plot of these functions should asymptotically become a straight line with the slope equal to the dynamical critical exponents. The local slopes for the survival probability $P(t)$ are defined by

$$\delta(t)=-\frac{\ln[P(t)/P(t/b)]}{\ln(b)} \tag{16}$$

and similarly for the other exponents. I usually use $b=15$. Away from criticality there are either upward or downward curvatures in the log-log plot of functions vs t and also in the plot of the critical exponents vs $1/t$, depending upon the super- or subcriticality of the system. By detecting the straight line from the curved lines, the value of τ_c can be evaluated with good precision. Having τ_c, the critical exponents can also be found.

Simulations are typically done up to time 400 (although many runs enter the absorbing state earlier) with a time increment of 0.004 for continuous-time simulation. Obviously, this is equal to the maximum precision possible in determining τ_c. Statistical quantities have been generally averaged over 10 000 different realizations of the model.

Figure 3 shows the results of the dynamical simulations. In different panels local slopes as defined in Eq. (16) for different dynamical critical exponents have been depicted against $1/t$. The best estimation for the critical age based on these graphs is $\tau_c \simeq 3.07(1)$. For the critical exponents I found $\eta = 0.304(1)$, $\delta = 0.1653(1)$, and $z = 1.272(1)$. These critical exponents are in good agreement with those of DP and thus the system belongs to the DP class.

VI. CONCLUSION

In summary, a non-Markovian version of the contact process is introduced. Particles have an age which describes when they are born and determines when they will be annihilated. Interesting oscillatory behaviors are observed in density of particles and the same oscillations are also present in the mean-field approximation. Because of the non-Markovian property of the model there is a time delay in the mean-field first-order differential equation which allows it to demonstrate oscillatory behaviors.

Applying a time-dependent Monte Carlo technique, critical properties of the absorbing phase transition have been investigated and shown to belong to the DP universality class. The DP class is extended to contain a non-Markovian model.

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